Introduction

* Splines are everywhere – in the design of furniture, any engineered object. Toy story was the first movie with characters made out of splines. CEO of Disney, Ed Catmull.
* Fonts on a computer - CAGD
* Subdivision curve -- How can you take a bunch of jagged points connected by lines, and transform it into a smooth curve? Keep dividing and then removing the corners, and in the limit you end up with a spline.
* Spline was a physical device, a bar pinned down at certain points, but bent. And these curves resemble piecewise polynomials.
* Splines also appear in statistics with the central limit theorem:

Adding two or more of these random variables X in [0,1] gives spline

* And also in finite element methods in ODEs, PDEs
* Splines can be used to get rid of oscillations that polynomial interpretation gives rise to
* Splines = Piecewise polynomial function

Definitions

Let represent a partition of the in interval [a,b]:

The points in the partition are called breakpoints or also:

Spline space

Examples:

* looks like a piecewise constant function that can have jump discontinuities. A step function.
* on the other hand looks like continuous streak of line segments
* , which is not interesting from the point of splines

Natural Cubic Splines (k=3)

How to find a curve that passes through 4 points, that is smooth without any artificial curves.

Before the theory, engineers used physical devices, splines to bend material in the curve they wanted.

Mathematically this turns into a “variational problem”:

Find a function:

s.t it interpolates the points, and has the smallest curvature:

Where is the curvature of at x.

This turns into a nonlinear problem, which can be made simpler using an approximation of curvature as:

Then this turns into a solvable problem, with a solution of a cubic spline:

Where:

1. The spline interpolates the data points
2. The integral of curvature for every other function that is twice differentiable over the interval is greater than the spline

Proof.

We can write as the sum of two functions:

,

Then integrating the curvature

Cubic splines converge

You can rephrase this technique for regression:

Look at the book to find out how to program the .

Algorithm

**Picture**

Picturing a natural cubic spline , it passes through points called knots. On the intervals between each knot, the spline is a a cubic polynomial .

**Properties of Natural Cubic Splines**

1. Natural boundary conditions arise from minimizing curvature:
2. Interpolation
3. Continuity Conditions (1st and 2nd derivatives are continuous)

**Algorithm**

The idea is that first find the values , and once you find those values, those uniquely determine the spline. These ’s are called auxiliary parameters.

Suppose we know the and note that ,

Integrating twice, we get an expression for our polynomials:

So in each interval we have two unknowns , and two interpolation conditions:

The last thing to make sure is that the first derivatives are equal:

This with some algebra leads to a tridiagonal system…

Truncated Power Series

Dimension of Spline Space that has knots :

This is because you can construct a basis for this space:

The right side of the basis are truncated power functions:

Truncated power functions are differentiable k-1 times.